Resolution of singularities of null cones*

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Abstract

We give canonical resolutions of singularities of several cone varieties arising from invariant theory. We establish a connection between our resolutions and resolutions of singularities of closure of conjugacy classes in classical Lie algebras.

Introduction

The purpose of this paper is to present canonical resolutions of singularities of certain cone varieties arising naturally in invariant theory. Examples of such cone varieties are provided by the so-called null cones which are the zero locus of the O_n (or Sp_n) invariant polynomials on $M_{n,m}$, where $M_{n,m}$ denotes the space of $n \times m$ complex matrices (cf. e.g. [H]). Another example is the null cone for general linear groups. Our construction resembles the celebrated Springer resolution of the nilpotent variety in a complex semisimple Lie algebra which plays a vital role in representation theory and geometry (cf. [CG]).

It turns out that the cone varieties studied here include as special cases the variety Z constructed in [KP1, KP2] associated to two-column partitions. The Kraft-Procesi variety Z is a complete intersection whose quotient under certain group coincide with the closure of a nilpotent conjugacy class in classical Lie algebras. It is a very interesting problem to construct a canonical resolution of singularities of Z in general. We establish relations between our resolutions of singularities and those for closure of conjugacy classes associated to two-column partitions. In the general linear Lie algebra case, we find more than one canonical resolution of singularities of null cones and of closure of conjugacy classes.

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We mention in passing that the present work grows out of an attempt to generalize the geometric construction of (gl_n, gl_m)-duality [W] in the spirit of [CG] to other Howe duality [H].

The paper is divided into two sections. In Sect. 1 we present resolutions of singularities of null cones in the O_n and Sp_n cases, and establish connections with resolutions of singularities of closure of conjugacy classes. In Sect. 2 we treat the analog of Sect. 1 for the general linear Lie algebras.

1 Null cones associated to O_n and Sp_n

Denote by GL_n, O_n, and Sp_{2n} the complex general linear group, orthogonal group, and the symplectic group respectively. Denote by $M_{n,m}$ the set of all complex $n \times m$ matrices.

1.1 The orthogonal setup

Let V be a vector space of complex dimension n with a non-degenerate symmetric bilinear form. We identify V with $\mathbb{C}^n$ endowed with the standard symmetric bilinear form

$$(u, v) = \sum_{i=1}^{n} u_i v_i,$$

where u, v denote the n-tuples $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$.

For convenience, we can identify $M_{n,m}$ with the direct sum $(\mathbb{C}^n)^m$ of m copies of $\mathbb{C}^n$. We may write a typical element of $(\mathbb{C}^n)^m$ as an m-tuple:

$$A = (v_1, v_2, \ldots, v_m), \quad v_j \in \mathbb{C}^n.$$

We define a set of quadratic polynomials $\tilde{\xi}_{ij} = \tilde{\xi}_{ji}, 1 \leq i, j \leq m$ on $(\mathbb{C}^n)^m$ by

$$\tilde{\xi}_{ij}(v) = (v_i, v_j) = \sum_{a=1}^{n} v_{ai} v_{aj}.$$

Remark 1 The First Fundamental Theorem for O_n (cf. [H]) says that the polynomials $\tilde{\xi}_{ij} = \tilde{\xi}_{ji}, 1 \leq i, j \leq m$, generate the algebra of O_n invariant polynomials on $(\mathbb{C}^n)^m$.

We identify the space $S^2(\mathbb{C}^m)$ of second symmetric tensors as the space of symmetric $m \times m$ matrices. Define a map

$$\begin{array}{rcl} Q : M_{n,m} & \longrightarrow & S^2(\mathbb{C}^m) \\ T & \mapsto & T^t T. \end{array}$$

Here T^t denotes the transpose of T. By identifying $M_{n,m}$ with $(\mathbb{C}^n)^m$, we can equivalently define $Q(T)$ as the $m \times m$ symmetric matrix whose (i, j)-th entry is equal to $\tilde{\xi}_{ij}$.

1.2 The symplectic setup

Let V be a vector space of complex dimension n with a non-degenerate symplectic (i.e. anti-symmetric) bilinear form. It is well known that n has to be an even number, say $2p$. We identify V with $\mathbb{C}^n$ endowed with the following standard symplectic bilinear form

$$(v, v') = \sum_{j=1}^{n/2} (x_j y'_j - y_j x'_j).$$

Here $v = (x, y)$ and $v' = (x', y')$ are elements of $\mathbb{C}^{2p}$, expressed as 2-tuples of elements of $\mathbb{C}^p$.

We identify $M_{n,m}$ with the direct sum $(\mathbb{C}^n)^m$ of m copies of $\mathbb{C}^n$ as before. We may write a typical element of $(\mathbb{C}^n)^m$ as an m-tuple:

$$A = (v_1, v_2, \ldots, v_m), \quad v_j \in \mathbb{C}^n.$$

We define a set of quadratic polynomials $\check{\xi}_{ij} = -\check{\xi}_{ji}, 1 \leq i, j \leq m$ on $(\mathbb{C}^n)^m$ by

$$\tilde{\xi}_{ij}(v) = (v_i, v_j).$$

Remark 2 The First Fundamental Theorem for Sp_n (cf. [H]) says that the polynomials $\check{\xi}_{ij} = -\check{\xi}_{ji}, 1 \leq i, j \leq m$, generate the algebra of Sp_n invariant polynomials on $(\mathbb{C}^n)^m$.

We identify the space $\Lambda^2(\mathbb{C}^m)$ of second anti-symmetric tensors in $\mathbb{C}^m$ as the space of anti-symmetric $m \times m$ matrices. Denote by

$$J = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix},$$

where I_p denotes the identity $p \times p$ matrix. Define a map

$$\begin{array}{rcl} Q_{sp} : M_{n,m} & \longrightarrow & \Lambda^2(\mathbb{C}^m) \\ T & \mapsto & T^t J T. \end{array}$$

We also write Q_{sp} as Q when no ambiguity arises. By identifying $M_{n,m}$ with $(\mathbb{C}^n)^m$, we can equivalently define $Q(T)$ as the $m \times m$ anti-symmetric matrix whose (i, j)-th entry equal to $\check{\xi}_{ij}$.

1.3 A resolution of singularities of $\mathcal{NCQ}$

Let V be a vector space of complex dimension n with a nondegenerate bilinear form, either symmetric or anti-symmetric. Let $G(V)$ be the isometry group of the

form, which is O_n in symmetric case and Sp_n in anti-symmetric case. Let $\mathfrak{g}(V)$ be its Lie algebra.

Given A in $S^2(\mathbb{C}^m)$ (resp. $\Lambda^2(\mathbb{C}^m)$), its inverse image $Q^{-1}(A)$ under the map Q is referred to as the fiber of Q over A. Of particular interest is the fiber over 0 (cf. [H]), which we will refer to as the *null cone* (since it is obviously preserved by scalar dilations) and denote by $\mathcal{NCQ}$. Equivalently, the variety $\mathcal{NCQ}$ is the set of $n \times m$ matrices on which all $G(V)$ invariant polynomials on $M_{n,m}$ take value 0. Often we will think of $M_{n,m}$ as the space $\mathrm{Hom}(\mathbb{C}^m, V)$ of all linear maps from $\mathbb{C}^m$ to V (or $\mathbb{C}^n$). An element in $\mathcal{NCQ}$ is called a *null mapping*. We easily have

$$Q(gT) = Q(T), \quad T \in M_{n,m},\ g \in G(V), \tag{1}$$
$$Q(Th) = h^t Q(T) h, \quad h \in GL_m. \tag{2}$$

Remark 3 It follows from Eqs. (1) and (2) that $G(V)$ acts on $\mathcal{NCQ}$ and this action commutes with the action of GL_m. The space of regular functions on $\mathcal{NCQ}$, under the induced action of $G(V) \times GL_m$, has a beautiful decomposition into $G(V) \times GL_m$-modules (cf. [H]).

A subspace of V is called *isotropic* if any two vectors in the subspace are orthogonal to each other with respect to the corresponding bilinear form. We observe that $T \in M_{n,m} = \mathrm{Hom}(\mathbb{C}^m, V)$ is a null mapping if and only if the image of T, denoted by $\Im T$, form an isotropic subspace of $\mathbb{C}^n$. Denote by $J_r(V)$ the set of all r-dimensional isotropic subspaces of V. The set $J_r(V)$ is nonempty if and only if $r \le n/2$. It is well known that $G(V)$ acts on $J_r(V)$ transitively (cf. [H], Appendix 3). The dimension of $J_r(V)$ ($r \le n/2$) can be calculated to be $r(2n - 3r - 1)/2$.

Note that the null cone $\mathcal{NCQ}$ is a singular variety defined in terms of a finite set of quadratic equations. The first goal of this paper is to present a canonical resolution of singularities of $\mathcal{NCQ}$. Our construction is reminiscent of the Springer resolution of singularities of the nilpotent cone in a complex semisimple Lie algebra (cf. e.g. [CG]).

Set $r = \min(m, [n/2])$ from now on, where $[n/2]$ denotes the integer closest to and no larger than $n/2$. We introduce the following variety

$$\widetilde{\mathcal{NCQ}} = \{(T, U) \in \mathcal{NCQ} \times J_r(V) \mid \Im T \subset U\}.$$

We have the following projection maps:

$$\begin{array}{ccc} & \widetilde{\mathcal{NCQ}} & \\ \mu_0 \swarrow & & \searrow \pi_0 \\ \mathcal{NCQ} & & J_r(V) \end{array}$$

The diagram is $O_n \times GL_m$-equivariant, where GL_m acts trivially on $\widetilde{\mathcal{NCQ}}$ and $J_r(V)$. μ_0 is proper since $\widetilde{\mathcal{NCQ}} \subset \mathcal{NCQ} \times J_r(V)$ and $J_r(V)$ is compact. It is easy to see that π_0 and μ_0 are both surjective.

Denote by T_J the tautological bundle over $J_r(V)$:

$$T_J = \{(u, U) \in V \times J_r(V) \mid u \in U\}.$$

Denote by $\underline{\mathbb{C}}^r$ the rank r trivial bundle over $J_r(V)$. Given $U \in J_r(V)$, the fiber of π_0 over U is canonically identified with $\mathrm{Hom}(\mathbb{C}^m, U)$. Thus we have

$$\begin{aligned} \dim \widetilde{\mathcal{NCQ}} &= mr + \dim J_r(V) \\ &= mr + r(2n - 3r - 1)/2 \\ &= r(2m + 2n - 3r - 1)/2. \end{aligned}$$

This proves the following characterization of $\widetilde{\mathcal{NCQ}}$.

Proposition 1 *The variety $\widetilde{\mathcal{NCQ}}$ is isomorphic to the tensor product $Taut \otimes \underline{\mathbb{C}}^m$ of vector bundles $Taut$ and $\underline{\mathbb{C}}^m$ over the variety $J_r(V)$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{\mathcal{NCQ}} \arrow[rr, phantom, "\cong"] \arrow[dr, "\pi_0"'] & & T_J \bigotimes \underline{\mathbb{C}}^m \arrow[dl] \\ & J_r(V) & \end{tikzcd}$$

In particular, $\widetilde{\mathcal{NCQ}}$ is a smooth variety of dimension $r(2m+2n-3r-1)/2$, where $r = \min(m, [n/2])$.

Theorem 1 *The map $\mu_0 : \widetilde{\mathcal{NCQ}} \longrightarrow \mathcal{NCQ}$ is a resolution of singularities.*

Proof. It is clear that the set $\mathcal{NCQ}_0$ of all null mappings of maximal rank which is equal to $r = \min(m, [n/2])$ is a Zariski-open subvariety of $\mathcal{NCQ}$. Given $T \in \mathcal{NCQ}_0$ there exists a unique $U \in J_r(V)$ containing $\Im T$, namely $\Im T$ itself. Thus the map $\mu_0 : \widetilde{\mathcal{NCQ}} \longrightarrow \mathcal{NCQ}$ over an open set $\mu_0^{-1}(\mathcal{NCQ}_0)$ is one-to-one. Together with the smoothness of $\widetilde{\mathcal{NCQ}}$ provided by Proposition 1, we conclude the proof. $\square$

Remark 4 In the case $n \geq 2m$ and so $r = m$, we easily see that the map Q maps surjectively to the space Sym_m of $m \times m$ symmetric matrices, and it is submersive at any point T in $\mathcal{NCQ}_0$. It follows that

$$\begin{aligned} \dim \mathcal{NCQ} &= \dim M_{n,m} - \dim Sym_m \\ &= nm - (m^2 + m)/2 = \dim \widetilde{\mathcal{NCQ}}. \end{aligned}$$

This of course was also implied by Theorem 1.

It is known that $J_r(V)$ is disconnected if and only if we are in the orthogonal case and $n = 2m$ (which implies $r = m$) (cf. [H], Appendix 2). Let us assume that we are in such a case first of all, and so $J_r(V)$ has two smooth connected

components (cf. [H]). It follows by Proposition 1 that $\widetilde{\mathcal{NCQ}}$ also has two smooth connected components, denoted by $\widetilde{\mathcal{NCQ}}^+$ and $\widetilde{\mathcal{NCQ}}^-$ respectively. The null cone $\mathcal{NCQ}$ also consists of two irreducible components $\mathcal{NCQ}^{\pm}$, which are the image of the two connected components of $\widetilde{\mathcal{NCQ}}$ respectively.

Now assume we are in the symplectic case, or in the orthogonal case but $n \neq 2m$. Then $J_r(V)$ is connected smooth and so is $\widetilde{\mathcal{NCQ}}$. Thus $\mathcal{NCQ}$ is irreducible as the image of the surjective map μ_0 of the irreducible variety $\widetilde{\mathcal{NCQ}}$.

1.4 Relations with closure of conjugacy classes

In this subsection,we always assume that $n \geq 2m$, and in addition m is even in the orthogonal case. We need to reformulate the definition of $\mathcal{NCQ}$.

Let W be a vector space of complex dimension m with a non-degenerate bilinear form of type opposite to the one on V. We identify $\mathrm{Hom}(W, V) = M_{nm}$. Given any $T \in \mathrm{Hom}(W, V)$ the adjoint T^* is defined by

$$(Tw, v)_V = (w, T^* v)_W, \quad w \in W, v \in V.$$

Here we use the subscripts to indicate to which vector space a bilinear form belong.

Remark 5 In the setup of Subsect. 1.1 and 1.2 and $W = \mathbb{C}^m$, we easily check by definition that if the bilinear form on V is anti-symmetric then $T^* = -JT^t$; if the bilinear form on V is symmetric, then $T^* = T^t J$.

Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}(W, V) & \xrightarrow{\widetilde{Q}} & \mathfrak{g}(W) \\ \downarrow R & & \\ \mathfrak{g}(V) & & \end{array}$$

where $\widetilde{Q}$ is given by $T \mapsto T^*T$ and R by $T \mapsto TT^*$. It is easy to check that the images of $\widetilde{Q}$ and R lie in $\mathfrak{g}(W)$ and $\mathfrak{g}(V)$ respectively. It follows from definitions and Remark 5 that $\widetilde{Q}^{-1}(0) = Q^{-1}(0) \equiv \mathcal{NCQ}$.

Denote by $\mathcal{C}_\lambda$ the conjugacy class of the group $G(V)$ associated to the partition $\lambda = (2^m, 1^{n-2m})$, which is the intersection of $\mathfrak{g}(V)$ with the conjugacy class of $gl(V)$ associated to λ. The closure $\overline{\mathcal{C}}_\lambda$ is indeed the variety of endomorphisms in $\mathfrak{g}(V)$ of rank no greater than m and whose image is an isotropic subpace of V.

The variety $\mathcal{NCQ} = \widetilde{Q}^{-1}(0)$ appears as a special case of the variety Z in [KP2]. Recall that a quotient of a G-variety M by the group G is by definition the spectrum of the algebra of G-invariant regular functions on M. A special case of a theorem of Kraft and Procesi relevant to our considerations can be formulated as follows.

Theorem 2 *The map R maps $\mathcal{NCQ}$ surjectively to the closure $\overline{\mathcal{C}}_\lambda$ of the conjugacy class $\mathcal{C}_\lambda$ associated to the partition $\lambda = (2^m, 1^{n-2m})$. Furthermore R can be identified with the quotient map by $G(W)$ from $\mathcal{NCQ}$ to $\overline{\mathcal{C}}_\lambda$.*

Define the variety

$$\widetilde{\mathcal{C}}_\lambda = \{(g, U) \in \overline{\mathcal{C}}_\lambda \times J_m(V) | \Im g \subset U\},$$

and denote by p_0 the (surjective) projection from $\widetilde{\mathcal{C}}_\lambda$ to the first factor $\overline{\mathcal{C}}_\lambda$. We can identify $\widetilde{\mathcal{C}}_\lambda$ with a vector bundle over $J_m(V)$, whose fiber over $U \in J_m(V)$ is the vector space

$$F_0 = \{g \in \mathfrak{g}(V) | \Im g \subset U\}.$$

We remark that for $g \in F_0$ we have $U \subset \ker g$ and thus $g^2 = 0 \in gl(V)$. Since $p_0 : \widetilde{\mathcal{C}}_\lambda \to \overline{\mathcal{C}}_\lambda$ is one-to-one over the open subset $p_0^{-1}(\mathcal{C}_\lambda)$ consisting of the endomorphisms of maximal rank m, we have proved that

Proposition 2 *The map $p_0 : \widetilde{\mathcal{C}}_\lambda \to \mathcal{C}_\lambda$ is a resolution of singularities.*

The quotient map R by $G(W)$ induces a natural quotient map by $G(W)$ from $\widetilde{\mathcal{NCQ}}$ to $\widetilde{\mathcal{C}}_\lambda$. We have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{NCQ}} & \xrightarrow{\mu_0} & \mathcal{NCQ} \\ \downarrow /G(W) & & \downarrow /G(W) \\ \widetilde{\mathcal{C}}_\lambda & \xrightarrow{p_0} & \overline{\mathcal{C}}_\lambda \end{array}$$

Remark 6 In the case $n = 2m$ and $G(V) = O_n$, the O_n-conjugacy class $\mathcal{C}_\lambda$ associated to $\lambda = (2^m)$ is *very even*, and so $\widetilde{\mathcal{C}}_\lambda$ splits into two SO_n-conjugacy class $\widetilde{\mathcal{C}}_\lambda^\pm$ (cf. [KP2]). Then the above diagram can be refined as follow:

$$\begin{array}{ccc} \widetilde{\mathcal{NCQ}}^\pm & \xrightarrow{\mu_0} & \mathcal{NCQ}^\pm \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{C}}_\lambda^\pm & \xrightarrow{p_0} & \mathcal{C}_\lambda^\pm \end{array}$$

where $\widetilde{\mathcal{C}}_\lambda^\pm$ is defined by

$$\widetilde{\mathcal{C}}_\lambda = \{(g, U) \in \overline{\mathcal{C}}_\lambda \times J_m^\pm(V) | \Im g \subset U\},$$

and $J_m^\pm(V)$ are the two connected components of $J_m(V)$.

2 Null cones associated to GL_n

Let V be a vector space of complex dimension n. Define a map

$$\varphi : \mathrm{Hom}(V, \mathbb{C}^s) \times \mathrm{Hom}(\mathbb{C}^m, V) \to \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^s) = M_{s,m}$$

by $\varphi(A, B) = AB = (\xi_{ij})_{1\le i\le s, 1\le j\le m}$. Hence ξ_{ij} defines a polynomial function on $\mathrm{Hom}(V, \mathbb{C}^s) \times \mathrm{Hom}(\mathbb{C}^m, V) \cong V^m \oplus (V^*)^s$. Denote by $\mathcal{N} = \varphi^{-1}(0)$. The group $GL_s \times GL(V) \times GL_m$ acts on $\mathcal{N}$ by

$$(g_1, g_2, g_3).(A, B) = (g_1 A g_2^{-1}, g_2 B g_3^{-1}).$$

Remark 7 The First Fundamental Theorem for $GL(V)$ says that the $\xi_{ij}, 1 \leq i \leq s, 1 \leq j \leq m$ generate the algebra of $GL(V)$ invariant polynomials on $V^m \oplus (V^*)^s$.

From now on we restrict ourselves to the case $n \geq s + m$.

2.1 A resolution of singularities of $\mathcal{N}$

We introduce a variety

$$\begin{aligned} \widetilde{\mathcal{N}} &= \{(A, B, U_1, U_2) \in \mathcal{N} \times \mathcal{F}(m, n-s, V) \mid \Im B \subset U_1 \subset U_2 \subset \ker A\} \\ &\subset \mathcal{N} \times \mathcal{F}(m, n-s, V), \end{aligned}$$

where $\ker A$ denotes the kernel of A, and $\mathcal{F}(m, n-s, V)$ denotes the generalized flag variety

$$\mathcal{F}(m, n-s, V) = \{(U_1, U_2) \mid U_1 \subset U_2 \subset V, \dim U_1 = m, \dim U_2 = n - s\}.$$

The group $GL_s \times GL(V) \times GL_m$ acts on $\widetilde{\mathcal{N}}$ by letting

$$(g_1, g_2, g_3).(A, B, U_1, U_2) = (g_1 A g_2^{-1}, g_2 B g_3^{-1}, g_2 U_1, g_2 U_2).$$

We have the following projection maps:

$$\begin{array}{ccc} & \widetilde{\mathcal{N}} & \\ \mu \swarrow & & \searrow \pi \\ \mathcal{N} & & \mathcal{F}(m, n-s, V) \end{array}$$

The diagram above is $GL_s \times GL(V) \times GL_m$-equivariant, where $G(V)$ acts naturally on the generalized flag variety $\mathcal{F}(m, n-s, V)$ while GL_s and GL_m act on $\mathcal{F}(m, n-s, V)$ trivially. It is easy to see that π and μ are surjective thanks to the assumption $n \geq s + m$, and that μ is a proper map due to the compactness of $\mathcal{F}(m, n-s, V)$. The fiber of π over a point $(U_1, U_2) \in \mathcal{F}(m, n-s, V)$ can be identified with the vector space

$$F = \mathrm{Hom}(V/U_2, \mathbb{C}^s) \times \mathrm{Hom}(\mathbb{C}^m, U_1). \tag{3}$$

In other words, $\widetilde{\mathcal{N}}$ can be identified with a vector bundle $\mathcal{K}$ over the generalized flag variety $\mathcal{F}(m, n-s, V)$:

$$\mathcal{K} := \underline{\mathbb{C}^m} \bigotimes \mathrm{Taut} \bigoplus \underline{\mathbb{C}^s} \bigotimes \mathcal{Q}^*,$$

where $\underline{\mathbb{C}^s}, \underline{\mathbb{C}^m}$ are trivial bundles of rank s and m respectively, and Taut, $\mathcal{Q}^*$ are respectively the tautological bundle and the dual quotient bundle:

$$\begin{aligned} \mathrm{Taut} &= \{(u, U_1, U_2) \mid v \in U_1\} \subset V \times \mathcal{F}(m, n-s, V) \\ \mathcal{Q}^* &= \{(v, U_1, U_2) \mid v \in (V/U_2)^*\} \subset V^* \times \mathcal{F}(m, n-s, V). \end{aligned}$$

Summarizing, we have proved that

Proposition 3 *We have the following isomorphism of vector bundles over the generalized flag variety $\mathcal{F}(m, n-s, V)$:*

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[dr, "\pi"'] & \cong & \mathcal{K} \arrow[dl] \\ & \mathcal{F}(m, n-s, V) & \end{tikzcd}$$

In particular, $\widetilde{\mathcal{N}}$ is a smooth variety of dimension $sn + mn - sm$.

Proof. It remains to calculate that

$$\begin{aligned} \dim \widetilde{\mathcal{N}} &= \dim \mathcal{F}(m, n-s, V) + \dim F \\ &= \frac{1}{2}(n^2 - (s^2 + (n-s-m)^2 + m^2) + (s^2 + m^2) \\ &= sn + mn - sm, \end{aligned}$$

where the equality $\dim \mathcal{F}(m, n-s, V) = \frac{1}{2}(n^2 - (s^2 + (n-s-m)^2 + m^2)$ follows readily from the description of the generalized flag variety $\mathcal{F}(m, n-s, V)$ in terms of the quotient of $GL(V)$ by an appropriate parabolic subgroup. $\square$

Theorem 3 *$\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities.*

Proof. The subset $\mathcal{N}_0$ of $\mathcal{N}$ consisting of pairs of maximal rank matrices (A, B) is a Zariski-open set in $\mathcal{N}$. A pair $(A, B) \in \mathcal{N}_0$ uniquely determines a pair $(U_1, U_2) \in \mathcal{F}(m, n-s, V)$ such that $(A, B, U_1, U_2) \in \widetilde{\mathcal{N}}$, namely $U_1 = \Im B, U_2 = \ker A$. This shows that μ maps $\mu^{-1}(\mathcal{N}_0) \subset \widetilde{\mathcal{N}}$ bijectively to $\mathcal{N}_0$. Together with the smoothness of $\widetilde{\mathcal{N}}$ proved in Proposition 3, we have concluded the proof. $\square$

Remark 8 Indeed it is easy to see that $\mathcal{N}_0$ is a single $GL(V)$-orbit since $n \geq s+m$. An easy calculation shows that φ is submersive over any point in $\mathcal{N}_0$. So we have

$$\begin{aligned} \dim \mathcal{N} &= \dim \mathrm{Hom}(V, \mathbb{C}^s) + \dim \mathrm{Hom}(\mathbb{C}^m, V) - \dim M_{s,m} \\ &= sn + mn - sm = \dim \widetilde{\mathcal{N}} \end{aligned}$$

by Proposition 3. This fact, of course was implied by Theorem 3.

When $s = m$ and thus $n \geq 2m$, the space $\mathcal{N}$ appears as a special case of the variety Z studied in [KP1]. It is shown [KP1] that the quotient of $\mathcal{N}$ by the diagonal action of GL_m is the variety of $n \times n$ matrices of square 0 and rank at most m, which is the closure $\overline{\mathcal{O}}_\lambda$ of the conjugacy class $\mathcal{O}_\lambda \subset gl(V)$ corresponding to the partition $\lambda = (2^m, 1^{n-2m})$. More explicitly the quotient map from $\mathcal{N}$ to $\overline{\mathcal{O}}_\lambda$ is given by $(A, B) \mapsto BA$.

Denote by

$$\widetilde{\mathcal{O}}_\lambda = \{(g, U_1, U_2) \mid \Im g \subset U_1 \subset U_2 \subset \ker g\} \subset \overline{\mathcal{O}}_\lambda \times \mathcal{F}(m, n-m, V),$$

and by p the natural surjective projection from $\widetilde{\mathcal{O}}_\lambda$ to $\overline{\mathcal{O}}_\lambda$. By a similar argument which leads to Proposition 3, we have the following identification

$$\begin{array}{ccc} \widetilde{\mathcal{O}}_\lambda & \cong & Taut \bigotimes Q^* \\ \searrow & & \swarrow \\ & \mathcal{F}(m, n-m, V) & \end{array}$$

which implies that $\widetilde{\mathcal{O}}_\lambda$ is smooth. Since the natural surjective projection p from $\widetilde{\mathcal{O}}_\lambda$ to (the first factor) $\overline{\mathcal{O}}_\lambda$ is one-to-one over the open set $p^{-1}(\mathcal{O}_\lambda)$, it is a resolution of singularities. Thus we have established

Proposition 4 *The map $p : \widetilde{\mathcal{O}}_\lambda \longrightarrow \overline{\mathcal{O}}_\lambda$ is a resolution of singularities.*

Remark 9 This resolution $p : \widetilde{\mathcal{O}}_\lambda \longrightarrow \overline{\mathcal{O}}_\lambda$ differs from the classical one in terms of cotangent bundle of a grassmannian (compare with Proposition 6).

The quotient map from $\mathcal{N}$ to $\overline{\mathcal{O}}_\lambda$ induces a quotient map by GL_m from $\widetilde{\mathcal{N}}$ to $\widetilde{\mathcal{O}}_\lambda$ which makes the following digram

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \xrightarrow{\mu} & \mathcal{N} \\ \downarrow /GL_m & & \downarrow /GL_m \\ \widetilde{\mathcal{O}}_\lambda & \xrightarrow{p} & \overline{\mathcal{O}}_\lambda \end{array} \tag{4}$$

commutative.

2.2 A second resolution of singularities of $\mathcal{N}$

We introduce a variety

$$\begin{aligned} \widetilde{\mathcal{N}_1} &= \{(A, B, U) \in \mathcal{N} \times Gr(m, V) \mid \Im B \subset U \subset \ker A\} \\ &\subset \mathcal{N} \times Gr(m, V), \end{aligned}$$

where $Gr(m, V)$ denotes the grassmannian of m-dimensional subspaces of V. The group $GL_s \times GL(V) \times GL_m$ acts on $\widetilde{\mathcal{N}_1}$ by letting

$$(g_1, g_2, g_3).(A, B, U) = (g_1 A g_2^{-1}, g_2 B g_3^{-1}, g_2 U).$$

We have the following projection maps:

$$\begin{array}{ccc} & \widetilde{\mathcal{N}_1} & \\ \mu_1 \swarrow & & \searrow \pi_1 \\ \mathcal{N} & & Gr(m, V) \end{array}$$

The diagram above is $GL_s \times GL(V) \times GL_t$-equivariant, where $G(V)$ acts naturally on $Gr(m, V)$ while GL_s, GL_t act trivially on $Gr(m, V)$. It is easy to see that π_1

and μ_1 are surjective and that μ_1 is a proper map. The fiber of π_1 over a point $U \in Gr(m, V)$ can be identified with the vector space

$$F_1 = \mathrm{Hom}(V/U, \mathbb{C}^s) \times \mathrm{Hom}(\mathbb{C}^m, U).$$

In other words, $\widetilde{\mathcal{N}}_1$ can be identified with a vector bundle $\mathcal{K}_1$ over the $Gr(m, V)$:

$$\mathcal{K}_1 := \underline{\mathbb{C}^m} \bigotimes \mathrm{Taut}_1 \bigoplus \underline{\mathbb{C}^s} \bigotimes \mathcal{Q}_1^*,$$

where $\underline{\mathbb{C}^s}, \underline{\mathbb{C}^m}$ are trivial bundles of rank s and t respectively, and Taut_1, $\mathcal{Q}_1^*$ are respectively the following tautological bundle and the dual quotient bundle:

$$\begin{array}{rcl} \mathrm{Taut}_1 & = & \{(u, U) \mid v \in U\} \subset V \times Gr(m, V) \\ \mathcal{Q}_1^* & = & \{(v, U) \mid v \in (V/U)^*\} \subset V^* \times Gr(m, V). \end{array}$$

Summarizing, we have proved that

Proposition 5 *We have the following isomorphism of vector bundles over the generalized flag variety $Gr(m, V)$:*

$$\begin{array}{ccccc} \widetilde{\mathcal{N}}_1 & & \cong & & \mathcal{K}_1 \\ & \pi_1 \searrow & & \swarrow & \\ & & Gr(m, V) & & \end{array}$$

In particular, $\widetilde{\mathcal{N}}_1$ is a smooth variety of dimension $sn + mn - sm$.

Proof. It remains to calculate that

$$\begin{array}{rcl} \dim \widetilde{\mathcal{N}} & = & \dim Gr(m, V) + \dim F_1 \\ & = & m(n-m) + m^2(n-m)s \\ & = & sn + mn - sm. \end{array}$$

□

Using Proposition 5, the following theorem can now be proved in the same way as Theorem 3.

Theorem 4 *The map $\mu_1 : \widetilde{\mathcal{N}}_1 \to \mathcal{N}$ is a resolution of singularities.*

Now we restrict ourselves to the case $s = m$ and thus $n \geq 2m$. Recall that the Kraft-Procesi quotient map from $\mathcal{N}$ to $\overline{\mathcal{O}}_\lambda$ is given by $(A, B) \mapsto BA$. Denote by

$$\widetilde{\mathcal{O}}_\lambda^1 = \{(g, U) \mid \Im g \subset U \subset \ker g\} \subset \overline{\mathcal{O}}_\lambda \times Gr(m, V),$$

and by p_1 the surjective projection from $\widetilde{\mathcal{O}}_\lambda^1$ to the first factor $\overline{\mathcal{O}}_\lambda$. We easily have the following identification

$$\begin{array}{ccccc} \widetilde{\mathcal{O}}_\lambda^1 & & \cong & & T^*Gr(m, V) \\ & \searrow & & \swarrow & \\ & & Gr(m, V) & & \end{array}$$

where $T^*Gr(m,V)$ denotes the cotangent bundle over the grassmannian.

Noting that the fiber of the projection p_1 over $g \in \mathcal{O}_\lambda$ consists of a single point. Since $\widetilde{\mathcal{O}}^1_\lambda$ is smooth, the projection p_1 from $\widetilde{\mathcal{O}}^1_\lambda$ to $\overline{\mathcal{O}}_\lambda$ is a resolution of singularities. Thus we have established the following classical result.

Proposition 6 *The map $p_1 : \widetilde{\mathcal{O}}^1_\lambda \longrightarrow \overline{\mathcal{O}}_\lambda$ is a resolution of singularities.*

The quotient map from $\mathcal{N}$ to $\overline{\mathcal{O}}_\lambda$ induces a quotient map by GL_m from $\widetilde{\mathcal{N}}_1$ to $\widetilde{\mathcal{O}}^1_\lambda$ which makes the following digram

$$\begin{array}{lcl} \widetilde{\mathcal{N}}_1 & \xrightarrow{\mu_1} & \mathcal{N} \\ \downarrow /GL_m & & \downarrow /GL_m \\ \widetilde{\mathcal{O}}^1_\lambda & \xrightarrow{p_1} & \overline{\mathcal{O}}_\lambda \end{array}$$

commutative.

2.3 A third resolution of singularities of $\mathcal{N}$

We introduce the variety

$$\begin{aligned} \widetilde{\mathcal{N}}_2 &= \{(A,B,U) \in \mathcal{N} \times Gr(n-s,V) \mid \Im B \subset U \subset \ker A\} \\ &\subset \mathcal{N} \times Gr(n-s,V), \end{aligned}$$

together with the surjective projection μ_2 from $\widetilde{\mathcal{N}}_2$ to the first factor $\mathcal{N}$. The variety $\widetilde{\mathcal{N}}_2$ is a vector bundle over $Gr(n-s,V)$ of total dimension $sn+mn-sm$.

Theorem 5 *The map $\mu_2 : \widetilde{\mathcal{N}}_2 \longrightarrow \mathcal{N}$ is a resolution of singularities.*

Proofs of all the statements concerning $\widetilde{\mathcal{N}}_2$ and $\widetilde{\mathcal{O}}^2_\lambda$ are similar to those for $\widetilde{\mathcal{N}}_1$ and $\widetilde{\mathcal{O}}^1_\lambda$ in the previous subsection which we omit.

Now we restrict again to the case $s = m$ and $n \geq 2m$. Denote by

$$\widetilde{\mathcal{O}}^2_\lambda = \{(g,U) \mid \Im g \subset U \subset \ker g\} \subset \overline{\mathcal{O}}_\lambda \times Gr(n-m,V).$$

We again can identify $\widetilde{\mathcal{O}}^2_\lambda$ as the cotangent bundle $T^*Gr(n-m,V)$. We can show that the natural surjective projection p_2 from $\widetilde{\mathcal{O}}^2_\lambda$ to $\overline{\mathcal{O}}_\lambda$ is a resolution of singularities.

The quotient map from $\mathcal{N}$ to $\overline{\mathcal{O}}_\lambda$ induces a quotient map by GL_m from $\widetilde{\mathcal{N}}_2$ to $\widetilde{\mathcal{O}}^2_\lambda$ which makes the following digram

$$\begin{array}{lcl} \widetilde{\mathcal{N}}_2 & \xrightarrow{\mu_2} & \mathcal{N} \\ \downarrow /GL_m & & \downarrow /GL_m \\ \widetilde{\mathcal{O}}^2_\lambda & \xrightarrow{p_2} & \overline{\mathcal{O}}_\lambda \end{array}$$

commutative.

The relation among the three resolutions of $\mathcal{N}$ is shown by the following diagram:

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{N}} & \\
q_1 \swarrow & & \searrow q_2 \\
\widetilde{\mathcal{N}}_1 & & \widetilde{\mathcal{N}}_2 \\
\mu_1 \searrow & & \swarrow \mu_2 \\
 & \mathcal{N} &
\end{array}
$$

where the morphisms q_1 and q_2 are defined by sending (A, B, U_1, U_2) to (A, B, U_1) and (A, B, U_2) respectively.

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